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Polynomial ultradistributions on $\mathbb{R}_+^d$

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ABSTRACT

Let $\mathcal{G}'_+ = \mathcal{G}'(\mathbb{R}_+^d)$ stand for Roumieu ultradistributions with supports in the positive cone $\mathbb{R}_+^d$. Throughout $\mathcal{P}(\mathcal{G}'_+)$ denotes the algebra of continuous scalar polynomials on the space $\mathcal{G}'_+$. We investigate the dual pair $\langle \mathcal{P}'(\mathcal{G}'_+) \mid \mathcal{P}(\mathcal{G}'_+)\rangle$ generated by the algebra $\mathcal{P}(\mathcal{G}'_+)$ and by its strong dual $\mathcal{P}'(\mathcal{G}'_+)$. Properties of the polynomially extended operational calculus and the semigroups of shifts along the cone $\mathbb{R}_+^d$ are considered.

1. Introduction

We will denote by $\mathcal{G}'_+ = \mathcal{G}'(\mathbb{R}_+^d)$ the space of Roumieu ultradistributions which are supported by the positive cone $\mathbb{R}_+^d$. Let $\mathcal{P}(\mathcal{G}'_+)$ stand for the space of continuous scalar polynomials on the $\mathcal{G}'_+$. Our purpose is to investigate some properties of the topological algebra $\mathcal{P}(\mathcal{G}'_+)$ as well as properties of its strong dual $\mathcal{P}'(\mathcal{G}'_+)$. The algebra $\mathcal{G}'_+$ is topologically embedded in $\mathcal{P}'(\mathcal{G}'_+)$, hence $\mathcal{P}'(\mathcal{G}'_+)$ can be considered as a polynomial extension of the algebra of Roumieu ultradistributions.

We give a specification for the dual pair $\langle \mathcal{P}'(\mathcal{G}'_+) \mid \mathcal{P}(\mathcal{G}'_+)\rangle$ by means of the pair $\langle \mathsf{P}'(\mathcal{G}'_+) \mid \mathsf{P}(\mathcal{G}'_+)\rangle$ consisted of the corresponding convolution topological algebras of coefficients. It is showed for a general case of nuclear (F) and (DF) spaces in Section 2, which has a preliminary character.

Theorem 3.1 describes the generalized differentiations in $\mathcal{P}'(\mathcal{G}'_+)$ using the above-mentioned duality. Differentiation operators generate the semigroups of shifts along the cone $\mathbb{R}_+^d$. The convolution algebra $\mathcal{G}'_+$ can be isomorphically represented as the commutant of these semigroups, using the cross-correlation (see Theorem 4.1). Finally, in Section 5 we construct a polynomial extension of Laplace transformation and describe some of its properties.

Algebras of ultradistributions with the symmetric tensor operation of multiplication were used in physics (see e.g. [1]). It was an incitement to research the problems connected with the polynomially extended cross-correlation of ultradistributions and the corresponding operational calculus.

There are other known and widely used infinite-dimensional generalizations of classical distribution spaces which are based on modern Gaussian analysis methods as well as the concept of Gelfand triple (see e.g. [2–4]).

Note that problems connected with analyticity on classical spaces of distributions were investigated in [5].

2. A polynomial duality for nuclear (F) and (DF) spaces

For polynomial operators on vector spaces we refer to [6]. Let X, Y be locally convex (in short LC) complex vector spaces. We will denote by $\mathscr{L}(^nX, Y)$ the space of all continuous n-linear operators, which are defined on the Cartesian nth power $^nX := X \times \cdots \times X$ with the topology $\mathfrak{b}$ of uniform convergence on bounded sets in X. Further $\mathscr{L}_{\mathfrak{s}}(^nX, Y)$ stands for the set of

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symmetric n-linear operators. Write $\mathscr{L}(X,Y) := \mathscr{L}(^1X,Y)$ and $\mathscr{L}(X) := \mathscr{L}(X,X)$. Throughout $X' := \mathscr{L}(X,\mathbb{C})$ denotes the strong dual of X. We will denote: by $[[T]]$ the commutant of operator $T \in \mathscr{L}(X)$; by $\langle f \mid x\rangle$ the value of $f \in X'$ at $x \in X$; by $A' \in \mathscr{L}(Y',X')$ the adjoint operator of $A \in \mathscr{L}(X,Y)$.

In what follows, $\otimes_{\mathfrak{p}}$ (resp., $\odot_{\mathfrak{p}}$) denotes a completion of algebraic tensor product $\otimes$ (resp., symmetric tensor product $\odot$) in the projective tensor LC topology. Consider the projective tensor product $\otimes_{\mathfrak{p}}^n X'$ (resp., symmetric $\odot_{\mathfrak{p}}^n X'$) of n copies of the strong dual X'. The symmetrization projector

$$\mathfrak{s}_n\colon \otimes_{\mathfrak{p}}^n X' \ni f_1\otimes\cdots\otimes f_n \mapsto f_1\odot\cdots\odot f_n := \frac{1}{n!}\sum_{\mathfrak{s}} f_{\mathfrak{s}(1)}\otimes\cdots\otimes f_{\mathfrak{s}(n)} \in \odot_{\mathfrak{p}}^n X',$$

where the sum is taken over all permutations $\mathfrak{s}$ of the set $\{1,\ldots,n\}$, is continuous. Analogously, the projective tensor product $\otimes_{\mathfrak{p}}^n X$ and the symmetric product $\odot_{\mathfrak{p}}^n X$ may be considered for the space X. Further $T_1\otimes\cdots\otimes T_n$ and $\otimes^n T := T\otimes\cdots\otimes T$ with $T, T_j \in \mathscr{L}(X,Y)$ denotes the tensor product of operators, defined as $(T_1\otimes\cdots\otimes T_n)(x_1\otimes\cdots\otimes x_n) = T_1x_1\otimes\cdots\otimes T_nx_n$ with $x_j\in X$. Clearly, $T_1\otimes\cdots\otimes T_n \in \mathscr{L}(\otimes_{\mathfrak{p}}^n X, \otimes_{\mathfrak{p}}^n Y)$.

In what follows, $\times_{n\in\mathbb{Z}_+}(\odot_{\mathfrak{p}}^n X)$ denotes the LC Cartesian power and $\bigoplus_{n\in\mathbb{Z}_+}(\odot_{\mathfrak{p}}^n X)$ the LC direct sum of the symmetric products $\odot_{\mathfrak{p}}^n X$. Analogously, we may write such the Cartesian product and direct sum for the dual X'.

To define the LC space $\mathscr{P}_n(X)$ of n-homogeneous polynomials on X we use the canonical topological linear isomorphisms $\mathscr{P}_n(X) \approx \mathscr{L}_{\mathfrak{s}}(^nX,\mathbb{C}) \approx (\odot_{\mathfrak{p}}^n X)'$ described in [6]. Namely, consider the embeddings

$$\otimes_n\colon {}^nX \ni (x_1,\ldots,x_n) \longmapsto x_1\otimes\cdots\otimes x_n \in \otimes_{\mathfrak{p}}^n X,$$
$$\Delta_n\colon X\ni x \longmapsto (x,\ldots,x)\in {}^nX.$$

Then the isomorphism $(\odot_{\mathfrak{p}}^n X)' \ni p_n \longmapsto P_n := p_n\circ\otimes_n\circ\Delta_n \in \mathscr{P}_n(X)$ defines an n-homogeneous polynomial on X, as the composition

$$P_n(x) = p_n(\otimes^n x), \qquad \otimes^n x := x\otimes\cdots\otimes x = (\otimes_n\circ\Delta_n)x, \quad x\in X.$$

We equip $\mathscr{P}_n(X)$ with the topology $\mathfrak{b}$ of uniform convergence on bounded sets in X. Set $\mathscr{P}_0(X) = \mathbb{C}$. The space $\mathscr{P}(X)$ of continuous polynomials on X is defined as the complex linear span of all $\mathscr{P}_n(X)$ endowed with the topology $\mathfrak{b}$. The space $\mathscr{P}(X)$ is a topological algebra with the scalar unit $\mathbf{1}$ and the pointwise multiplication

$$P(x)\cdot Q(x) = \sum_{n\in\mathbb{Z}_+}\sum_{m=0}^{n} P_m(x)\cdot Q_{n-m}(x), \quad x\in X.$$

Let us denote the strong duals by $\mathscr{P}'(X)$, $\mathscr{P}_n'(X)$. We define similar spaces $\mathscr{P}(X')$, $\mathscr{P}_n(X')$ and their duals $\mathscr{P}'(X')$, $\mathscr{P}_n'(X')$ for X'.

From now on we will assume that X is a LC nuclear (F) or (DF) space.

Further, we will use the following known facts without a special mention. If X is a (F) space then its strong dual X' is a (DF) space, if X is a (DF) space then its strong dual X' is a (F) space. Each nuclear (F) or (DF) space X is reflexive [7, Th 4.4.12] and its strong dual X' is nuclear [8, Th 9.6], see also [17]. For (F) or (DF) spaces the nuclear property is preserved for subspaces, separable factor spaces, completions, countable direct sums, Cartesian products. Moreover, a LC space X is nuclear iff $X\otimes_{\mathfrak{p}} Y = X\otimes_{\mathfrak{e}} Y$ for any LC space Y [7, Th 5.4.1], where $\otimes_{\mathfrak{e}}$ denotes a completion in the injective tensor LC topology. If X, Y are nuclear then $X\otimes_{\mathfrak{p}} Y$ is also nuclear [8, Th 3.7.5] and the topological linear isomorphism $(X\otimes_{\mathfrak{p}} Y)/[\mathscr{N}(A)\otimes_{\mathfrak{p}}\mathscr{N}(B)] \approx [X/\mathscr{N}(A)]\otimes_{\mathfrak{p}}[Y/\mathscr{N}(B)]$ is true for any operators $A\in\mathscr{L}(X)$ and $B\in\mathscr{L}(Y)$ with the kernels $\mathscr{N}(A)$ and $\mathscr{N}(B)$ respectively [9, Th 3].

Proposition 2.1. *There exist the canonical linear topological surjective isomorphisms $\Upsilon_n^{X'}$, Υ_n^{X} and their linear extensions $\Upsilon_{X'}$, Υ_X*

$$\Upsilon_n^{X'}\colon \mathsf{P}_n(X) := \odot_{\mathfrak{p}}^n X' \ni f_n \longmapsto F_n := (f_n\circ\otimes_n\circ\Delta_n)\in\mathscr{P}_n(X),$$
$$\Upsilon_{X'}\colon \mathsf{P}'(X') := \bigtimes_{n\in\mathbb{Z}_+}(\odot_{\mathfrak{p}}^n X') \ni f = \bigtimes_{n\in\mathbb{Z}_+} f_n \longmapsto F = \bigtimes_{n\in\mathbb{Z}_+}\Upsilon_n^{X'}(f_n)\in\mathscr{P}'(X'),$$
$$\Upsilon_n^{X}\colon \mathsf{P}_n(X') := \odot_{\mathfrak{p}}^n X \ni q_n \longmapsto Q_n := (q_n\circ\otimes_n\circ\Delta_n)\in\mathscr{P}_n(X'),$$
$$\Upsilon_{X}\colon \mathsf{P}(X') := \bigoplus_{n\in\mathbb{Z}_+}(\odot_{\mathfrak{p}}^n X) \ni q = \bigoplus_{n\in\mathbb{Z}_+} q_n \longmapsto Q = \sum_{n\in\mathbb{Z}_+}\Upsilon_n^{X}(q_n)\in\mathscr{P}(X').$$

The spaces $\mathscr{P}_n(X)$, $\mathscr{P}'(X')$ are nuclear, reflexive and the following equalities for dualities are true

$$\langle\mathscr{P}_n(X)\mid\mathscr{P}_n(X')\rangle = \langle\odot_{\mathfrak{p}}^n X'\mid\odot_{\mathfrak{p}}^n X\rangle,$$
$$\big\langle\mathscr{P}'(X')\mid\mathscr{P}(X')\big\rangle = \left\langle \bigtimes_{n\in\mathbb{Z}_+}(\odot_{\mathfrak{p}}^n X') \,\middle|\, \bigoplus_{n\in\mathbb{Z}_+}(\odot_{\mathfrak{p}}^n X)\right\rangle.$$

Besides, $\Upsilon_n^{X'} = [(\Upsilon_n^{X})']^{-1}$, $\Upsilon_{X'} = [(\Upsilon_X)']^{-1}$.

If we suppose the continuous dense embedding $X \hookleftarrow X'$, then the following continuous dense embeddings are true

$$\mathcal{P}_n(X') \hookleftarrow \mathcal{P}_n(X), \qquad \mathcal{P}(X') \hookleftarrow \mathcal{P}'(X').$$

Proof. Using the topological isomorphism $\odot_{\mathrm{p}}^n X' \approx (\odot_{\mathrm{p}}^n X)'$, which is true for a nuclear (F) or (DF) space X [8, Th 9.9], we obtain the first isomorphism $\Upsilon_n^{X'}$. Therefore, the duality $\left\langle \bigtimes_n (\odot_{\mathrm{p}}^n X') \mid \bigoplus_n (\odot_{\mathrm{p}}^n X)\right\rangle$, defined by the formula

$$F(x) = \left\langle \bigtimes_{n\in\mathbb{Z}_+} f_n \middle| \bigoplus_{n\in\mathbb{Z}_+} (\otimes^n x) \right\rangle = \sum_{n\in\mathbb{Z}_+} F_n(x), \qquad F_n(x) = \langle f_n \mid \otimes^n x\rangle, \tag{1}$$

with $x \in X$, gives the following isomorphism

$$\Upsilon_{X'} \colon \bigtimes_{n\in\mathbb{Z}_+} (\odot_{\mathrm{p}}^n X') \ni f = \bigtimes_{n\in\mathbb{Z}_+} f_n \longmapsto F = \Upsilon_{X'}(f) = \bigtimes_n \Upsilon_n^{X'}(f_n) \in \mathcal{P}(X),$$

acting as a linear extension of the mapping $\Upsilon_n^{X'} \colon \odot_{\mathrm{p}}^n X' \longmapsto \mathcal{P}_n(X)$.

Replacing X by X' in the previous paragraph, we immediately obtain the topological isomorphism $\odot_{\mathrm{p}}^n X \overset{\Upsilon_n^X}{\approx} \mathcal{P}_n(X')$. Therefore, $\bigoplus_n (\odot_{\mathrm{p}}^n X) \overset{\Upsilon_X}{\approx} \mathcal{P}(X')$.

Applying the well-known [8, 4.4] duality $\left\langle \bigtimes_n (\odot_{\mathrm{p}}^n X') \mid \bigoplus_n (\odot_{\mathrm{p}}^n X)\right\rangle$, we obtain the isomorphisms

$$\mathcal{P}'(X') \approx \left[\bigoplus_{n\in\mathbb{Z}_+} (\odot_{\mathrm{p}}^n X)\right]' \approx \bigtimes_{n\in\mathbb{Z}_+} (\odot_{\mathrm{p}}^n X') \approx \bigtimes_{n\in\mathbb{Z}_+} \mathcal{P}_n(X).$$

Hence, the previous duality may be transformed to $\left\langle \mathcal{P}'(X') \mid \mathcal{P}(X')\right\rangle$.

The canonical embedding $\bigoplus_n (\odot_{\mathrm{p}}^n X') \subset \bigtimes_n (\odot_{\mathrm{p}}^n X')$ implies

$$\mathcal{P}(X) \approx \bigoplus_{n\in\mathbb{Z}_+} (\odot_{\mathrm{p}}^n X') \subset \bigtimes_{n\in\mathbb{Z}_+} (\odot_{\mathrm{p}}^n X') \approx \mathcal{P}'(X').$$

Using that $\bigoplus_n (\otimes^n x)$ is a total subset in $\bigoplus_n (\odot_{\mathrm{p}}^n X)$, the mapping Υ_X can be uniquely linearly extended to $\Upsilon_{X'}$ by means of the formula (1).

Let us suppose the dense continuous embedding $X \hookleftarrow X'$. Then the embeddings $\bigoplus_n (\odot_{\mathrm{p}}^n X) \hookleftarrow \bigoplus_n (\odot_{\mathrm{p}}^n X')$ and $\bigoplus_n (\odot_{\mathrm{p}}^n X') \hookleftarrow \bigtimes_n (\odot_{\mathrm{p}}^n X')$ imply that the following dense continuous embeddings

$$\mathcal{P}(X') \approx \bigoplus_{n\in\mathbb{Z}_+} (\odot_{\mathrm{p}}^n X) \hookleftarrow \bigoplus_{n\in\mathbb{Z}_+} (\odot_{\mathrm{p}}^n X') \hookleftarrow \bigtimes_{n\in\mathbb{Z}_+} (\odot_{\mathrm{p}}^n X') \approx \mathcal{P}'(X')$$

are also true. The rest follows from the previous remarks. □

Remark 2.2. It follows from (1) that elements of $\mathcal{P}'(X')$ can be interpreted as polynomials on a space X.

Proposition 2.3. (i) *The LC space $\mathcal{P}'(X')$ is a completion in the strong topology of a set of finite type polynomials $\sum_{n\in\mathbb{Z}_+} \sum_{f_j\in X'} \langle f_1 \otimes \cdots \otimes f_n \mid \otimes^n x\rangle$ (as functions of the variable $x \in X$). The dual space X' is closed in $\mathcal{P}'(X')$.*

(ii) *The LC space $\mathcal{P}(X')$ is a completion (with respect to the topology of uniform convergence on bounded sets in X') of a set of finite type polynomials $\sum_{n\in\mathbb{Z}_+} \sum_{x_j\in X} \langle \otimes^n f \mid x_1 \otimes \cdots \otimes x_n\rangle$ (as functions of the variable $f \in X'$). The space X is closed in $\mathcal{P}(X')$.*

(iii) *The direct sum $\mathsf{P}(X')$ with a dense subset of elements $q^{\prime,\prime\prime} = \bigoplus_n q_n^{\prime,\prime\prime}$, ($q_n^{\prime,\prime\prime} \in \odot_{\mathrm{p}}^n X$) is a LC algebra with respect to the convolution*

$$q' \star q'' := \bigoplus_{n\in\mathbb{Z}_+} \sum_{m=0}^{n} q'_m \odot q''_{n-m}$$

and the mapping $\Upsilon_X \colon \{\mathsf{P}(X'), \star\} \longrightarrow \{\mathcal{P}(X'), \cdot\}$ acts as an isomorphism between convolution and multiplicative algebras.

Proof. The statements (i) and (ii) follow from the isomorphisms $\mathcal{P}'(X') \approx \bigtimes_n (\odot_{\mathrm{p}}^n X')$ and $\mathcal{P}(X') \approx \bigoplus_n (\odot_{\mathrm{p}}^n X)$, established by Proposition 2.1, with the help of additional arguments that the spaces $\odot_{\mathrm{p}}^n X'$ and $\odot_{\mathrm{p}}^n X$ can be approximated by linear combinations of elements $f_1 \odot \cdots \odot f_n$ and $x_1 \odot \cdots \odot x_n$, respectively (see [9]). For (iii) it is enough to check up that the linear isomorphism Υ_X is also algebraic. For all $q'_n \in \odot_{\mathrm{p}}^n X$, $q''_k \in \odot_{\mathrm{p}}^k X$ we have $q'_n \odot q''_k \in (\odot_{\mathrm{p}}^n X) \odot (\odot_{\mathrm{p}}^k X) \subset \odot_{\mathrm{p}}^{n+k} X$. Hence, $q'_m \odot q''_{n-m} \in \odot_{\mathrm{p}}^n X$ and

$$\left\langle \otimes^m f \mid q'_m\right\rangle \cdot \left\langle \otimes^{n-m} f \mid q'_{n-m}\right\rangle = \left\langle \otimes^n f \mid q'_m \odot q''_{n-m}\right\rangle.$$

So, Υ_X is the required algebraic isomorphism. □

Let us suppose the continuous dense embedding $X \hookrightarrow X'$. Then the convolution in $\left\{\bigoplus_{n\in\mathbb{Z}_+}(\odot_{\mathfrak{p}}^n X), \star\right\}$ can be extended to the convolution

$$f \star g := \bigtimes_{n\in\mathbb{Z}_+}\left(\sum_{m=0}^{n} f_n \odot g_{n-m}\right)$$

in the Cartesian product $\bigtimes_{n\in\mathbb{Z}_+}(\odot_{\mathfrak{p}}^n X') = \left\{f = \bigtimes_n f_n : f_n \in \odot_{\mathfrak{p}}^n X'\right\}$, which also is a topological convolution algebra.

Proposition 2.4. *The multiplication in $\mathcal{P}(X')$ can be uniquely extended to the multiplication in $\mathcal{P}'(X')$, given by the formula*

$$(P \cdot Q)\left[\bigoplus_{n\in\mathbb{Z}_+}(\otimes^n x)\right] = \sum_{n\in\mathbb{Z}_+}\sum_{m=0}^{n} P_m(x)\cdot Q_{n-m}(x), \quad x \in X$$

with $P = \bigtimes_n P_n$, $Q = \bigtimes_n Q_n \in \mathcal{P}'(X')$, where $P_n, Q_n \in \mathcal{P}_n(X)$. Thus, $\mathcal{P}'(X')$ is a topological algebra and Υ_X uniquely extends to the following isomorphism between convolution and multiplicative algebras

$$\{\mathsf{P}'(X'), \star\} \overset{\Upsilon_{X'}}{\approx} \{\mathcal{P}'(X'), \cdot\}.$$

Proof. Proposition 2.1 together with Proposition 2.3 immediately imply that the extended mapping $\Upsilon_{X'} : \bigtimes_n(\odot_{\mathfrak{p}}^n X') \ni f = \bigtimes_n f_n \longmapsto F = \bigtimes_n F_n \in \mathcal{P}'(X')$ establishes the required isomorphism of algebras. □

We will consider the subalgebras of matrix diagonal operators of the form

$$\mathcal{L}_\Gamma\left[\mathsf{P}(X')\right] = \mathcal{L}\left[\mathsf{P}(X')\right]\bigcap\left[\begin{cases}\mathcal{L}\left[\mathsf{P}_n(X')\right] & : n = m\\ 0 & : n \neq m\end{cases}\right]_{n,m\in\mathbb{Z}_+}.$$

Using the isomorphism Υ_X we can identify the appropriate operator algebras, namely $\mathcal{L}[\mathsf{P}(X')] \approx \mathcal{L}[\mathcal{P}(X')]$, $\mathcal{L}_\Gamma[\mathsf{P}(X')] \approx \mathcal{L}_\Gamma[\mathcal{P}(X')]$. We denote the commutant in $\mathcal{L}_\Gamma(\cdot)$ of an operator $T \in \mathcal{L}_\Gamma(\cdot)$ by $[[T]]_\Gamma$ and use analogous notation for other isomorphisms and operator algebras.

3. A polynomial extension of Roumieu ultradistributions

Consider the d-dimensional positive cone $\mathbb{R}_+^d = [0,\infty)\times\cdots\times[0,\infty)$. Let $\operatorname{int}\mathbb{R}_+^d = (0,\infty)\times\cdots\times(0,\infty)$. On the set of vectors $\nu = (\nu_1,\ldots,\nu_d) \in \operatorname{int}\mathbb{R}_+^d$ we put the order $\{\nu' \prec \nu'' : \nu_1' < \nu_1'', \ldots, \nu_d' < \nu_d''\}$. For any $a = (a_1,\ldots,a_d)$, $b = (b_1,\ldots,b_d) \in \mathbb{R}^d$ let $[a,b] := [a_1,b_1]\times\cdots\times[a_d,b_d]$. Fix $\beta > 1$ and for any $\nu \in \operatorname{int}\mathbb{R}_+^d$ and $[a,b] \subset \mathbb{R}^d$ we define the Banach space $\mathcal{G}_{[a,b]}^{\nu}(\mathbb{R}^d)$ of complex smooth functions φ with $\operatorname{supp}\varphi \subseteq [a,b]$ and the norm

$$\|\varphi\|_{\mathcal{G}_{[a,b]}^{\nu}} := \sup_{\tau\in[a,b]}\sup_{k\in\mathbb{Z}_+^d}\frac{|\partial^k\varphi(\tau)|}{\nu^k k^{k\beta}},$$

with $k = (k_1,\ldots,k_d)$, $\tau = (\tau_1,\ldots,\tau_d)$, $k^{k\beta} = k_1^{k_1\beta}\cdot\ldots\cdot k_d^{k_d\beta}$, $\nu^k = \nu_1^{k_1}\cdot\ldots\cdot\nu_d^{k_d}$, $\partial^k = \partial_1^{k_1}\cdots\partial_d^{k_d}$, $\partial_l^{k_l} := (-\mathrm{i})^{k_l}\frac{\partial^{k_l}}{\partial^{k_l}\tau_l}$, $l = 1,\ldots,d$.

The space $\mathcal{G}(\mathbb{R}^d)$ of Gevrey ultradifferentiable functions on $\mathbb{R}^d$ with compact supports can be defined as the inductive limit

$$\mathcal{G}(\mathbb{R}^d) = \operatorname*{ind\,lim}_{-a,b,\nu\to\infty}\mathcal{G}_{[a,b]}^{\nu}(\mathbb{R}^d), \quad (\nu\to\infty \text{ iff } \nu_l\to\infty, \forall l;\ \text{similarly for } -a, b).$$

As it is well known [10–12], $\mathcal{G}(\mathbb{R}^d)$ is a nuclear (DF) space and is a topological algebra with respect to the pointwise multiplication.

Let us denote the strong dual of $\mathcal{G}(\mathbb{R}^d)$ by $\mathcal{G}'(\mathbb{R}^d)$. Elements of $\mathcal{G}'(\mathbb{R}^d)$ are called (see [13]) by Roumieu ultradistributions on $\mathbb{R}^d$. Let $\mathcal{G}'(\mathbb{R}_+^d)$ stand for the closed subspace in $\mathcal{G}'(\mathbb{R}^d)$ of ultradistributions with supports in $\mathbb{R}_+^n$. If $[\mathcal{G}'(\mathbb{R}_+^d)]^\perp$ denotes the orthogonal complement of $\mathcal{G}'(\mathbb{R}_+^d)$ by the duality $\langle\mathcal{G}'(\mathbb{R}^d) \mid \mathcal{G}(\mathbb{R}^d)\rangle$ then the factor space

$$\mathcal{G}(\mathbb{R}_+^d) := \mathcal{G}(\mathbb{R}^d)/[\mathcal{G}'(\mathbb{R}_+^d)]^\perp = \left\{\boldsymbol{\varphi} := \varphi + [\mathcal{G}'(\mathbb{R}_+^d)]^\perp : \varphi \in \mathcal{G}(\mathbb{R}^d)\right\}$$

is dual of $\mathcal{G}'(\mathbb{R}_+^d)$. The multiplication operator

$$\Theta : \mathcal{G}(\mathbb{R}^d) \ni \varphi \longmapsto \theta_{\mathbb{R}_+^d}\varphi \in \mathcal{G}'(\mathbb{R}_+^d),$$

where $\theta_{\mathbb{R}_+^d}$ stands for the characteristic function of cone $\mathbb{R}_+^d$, has the kernel $[\mathcal{G}'(\mathbb{R}_+^d)]^\perp$. So, for its codomain $\Theta[\mathcal{G}(\mathbb{R}^d)]$ the topological linear isomorphism

$$\mathcal{G}(\mathbb{R}_+^d) \approx \Theta[\mathcal{G}(\mathbb{R}^d)]$$

is true. Thus, any element $\boldsymbol{\varphi} \in \mathcal{G}(\mathbb{R}_+^d)$ can be interpreted as a regular ultradistribution, belonging to $\mathcal{G}'(\mathbb{R}_+^d)$.

From duality reasons it follows that $\mathcal{G}'(\mathbb{R}^d_+)$ is a nuclear (F) space and $\mathcal{G}(\mathbb{R}^d_+)$ is a nuclear (DF) space. As it is known [14], $\mathcal{G}'(\mathbb{R}^d_+)$ is a topological algebra with respect to the convolution

$$(f,g)\longmapsto f*g,\quad f,g\in\mathcal{G}'(\mathbb{R}^d)$$

with the convolution unit $\mathbb{R}^d\ni(\tau_1,\dots,\tau_d)\longmapsto\delta(\tau_1)\cdots\delta(\tau_d)$, where δ is the Dirac function on $\mathbb{R}$. Since $[\mathcal{G}'(\mathbb{R}^d_+)]^\perp$ is a closed ideal in $\mathcal{G}(\mathbb{R}^d)$, the factor space $\mathcal{G}(\mathbb{R}^d_+)$ is also a topological algebra and Θ is an algebraic homomorphism.

The ideal $[\mathcal{G}'(\mathbb{R}^d_+)]^\perp$ of the algebra $\mathcal{G}(\mathbb{R}^d)$ is invariant with respect to shifts along the $\mathbb{R}^d_+$, hence for any $l=1,\dots,d$ the 1-parameter family of operators

$$T_{t_l}\colon\boldsymbol{\varphi}(\tau_1,\dots,\tau_d)\longmapsto\Theta\boldsymbol{\varphi}(\tau_1,\dots,\tau_{l-1},\tau_l+t_l,\tau_{l+1},\dots,\tau_d),$$

with $t_l\geq 0$ and $(\tau_1,\dots,\tau_d)\in\mathbb{R}^d_+$ forms a semigroup $T_l\colon 0\leq t_l\longmapsto T_{t_l}$ on the factor algebra $\mathcal{G}(\mathbb{R}^d_+)$.

The ideal $[\mathcal{G}'(\mathbb{R}^d_+)]^\perp$ is also invariant with respect to the partial differentiations ∂_l $(l=1,\dots,d)$, hence ∂_l are defined on the factor algebra $\mathcal{G}(\mathbb{R}^d_+)$. Note that operator ∂_l is a generator of semigroup T_l.

Let $T'_l\colon 0\leq t\longmapsto T'_{t_l}$ and $\partial'_l=-\partial_l$ be the adjoint operators with respect to the duality $\langle\mathcal{G}'(\mathbb{R}^d_+)\mid\mathcal{G}(\mathbb{R}^d_+)\rangle$ for all $l=1,\dots,d$.

In what follows, we will use the following short notations

$$\mathcal{G}_+:=\mathcal{G}(\mathbb{R}^d_+)\quad\text{and}\quad\mathcal{G}'_+:=\mathcal{G}'(\mathbb{R}^d_+).$$

Note that $\mathcal{G}_+$ is a nuclear (DF) space and $\mathcal{G}'_+$ is a nuclear (F) space. Moreover, the dense continuous embedding $\mathcal{G}_+\looparrowright\mathcal{G}'_+$ is true. Hence, we can consider the space of polynomials $\mathcal{P}(\mathcal{G}'_+)$ and its dual $\mathcal{P}'(\mathcal{G}'_+)$ for which the dense continuous embedding $\mathcal{P}(\mathcal{G}'_+)\looparrowright\mathcal{P}'(\mathcal{G}'_+)$ also is true via Proposition 2.1. Clearly, if we put $X=\mathcal{G}_+$ and $X'=\mathcal{G}'_+$ then all assertions of the previous section in the case of pair $\langle\mathcal{P}(\mathcal{G}'_+)\mid\mathcal{P}'(\mathcal{G}'_+)\rangle$ are valid.

Theorem 3.1. (i) *The 1-parameter families $\Gamma(T'_l)\colon 0\leq t_l\longmapsto\Gamma(T'_{t_l})$ (with $l=1,\dots,d$) of linear operators on the convolution algebra $\mathsf{P}(\mathcal{G}'_+)\overset{\Upsilon_{\mathcal{G}_+}}{\approx}\mathcal{P}(\mathcal{G}'_+)$, which are defined as*

$$\left[\Upsilon_{\mathcal{G}_+}\Gamma(T'_{t_l})\Upsilon^{-1}_{\mathcal{G}_+}\right]\mathsf{Q}(f)=\mathsf{Q}(T'_{t_l}f),\qquad \mathsf{Q}=\sum_{n\in\mathbb{Z}_+}\mathsf{Q}_n\in\mathcal{P}(\mathcal{G}'_+),\quad f\in\mathcal{G}'_+$$

where $\mathsf{Q}_n=q_n\circ\otimes_n\circ\Delta_n$ and $\Upsilon^{-1}_{\mathcal{G}_+}\mathsf{Q}_n=q_n\in\odot^n_{\mathrm{p}}\mathcal{G}_+$ are equicontinuous C_0-semigroups of algebraic automorphisms.

Their generators $\mathrm{d}\Gamma(\partial'_l)$ belong to the subalgebra $\mathscr{L}_\Gamma[\mathsf{P}(\mathcal{G}'_+)]$ and on any element $q=\bigoplus_n q_n$ with $\Upsilon^{-1}_{\mathcal{G}_+}\mathsf{Q}=q$ act as

$$\mathrm{d}\Gamma(\partial'_l)q=\bigoplus_{n\in\mathbb{Z}_+}\sum_{j=1}^{n}{}^n_j\partial_l q_n,\qquad {}^n_j\partial_l:=\otimes^{j-1}1_+\otimes\partial_l\otimes^{n-j}1_+.$$

Here 1_+ denotes the identity operator in $\mathscr{L}(\mathcal{G}_+)$.

(ii) *The families $\Gamma(T_l)\colon 0\leq t\longmapsto\Gamma(T_{t_l})$ $(l=1,\dots,d)$ of linear operators on the convolution algebra $\mathsf{P}'(\mathcal{G}'_+)\overset{\Upsilon_{\mathcal{G}'_+}}{\approx}\mathcal{P}'(\mathcal{G}'_+)$, which are defined as*

$$\left[\Upsilon_{\mathcal{G}'_+}\Gamma(T_{t_l})\Upsilon^{-1}_{\mathcal{G}'_+}\right]P(\boldsymbol{\varphi})=P(T_{t_l}\boldsymbol{\varphi})\qquad P=\mathop{\times}_{n\in\mathbb{Z}_+}P_n\in\mathcal{P}'(\mathcal{G}'_+),\quad\boldsymbol{\varphi}\in\mathcal{G}_+$$

with $P_n=p_n\circ\otimes_n\circ\Delta_n$ and $\Upsilon^{-1}_{\mathcal{G}'_+}P_n=p_n\in\odot^n_{\mathrm{p}}\mathcal{G}'_+$, are equicontinuous C_0-semigroups of algebraic automorphisms. Their generators $\mathrm{d}\Gamma(\partial_l)$ belong to the subalgebra $\mathscr{L}_\Gamma\left[\mathsf{P}'(\mathcal{G}'_+)\right]$ and act as

$$\mathrm{d}\Gamma(\partial_l)p=-\mathop{\times}_{n\in\mathbb{Z}_+}\sum_{j=1}^{n}{}^n_j\partial'_l p_n,\qquad p=\mathop{\times}_{n\in\mathbb{Z}_+}p_n,\qquad {}^n_j\partial'_l:=\otimes^{j-1}1'_+\otimes\partial'_l\otimes^{n-j}1'_+.$$

Here $1'_+$ denotes the identity operator in $\mathscr{L}(\mathcal{G}'_+)$.

(iii) *The generators $\mathrm{d}\Gamma(\partial'_l)$ $(l=1,\dots,d)$ are continuous differentiations on the convolution algebra $\mathsf{P}(\mathcal{G}'_+)$, i.e.,*

$$\mathrm{d}\Gamma(\partial'_l)(p\star q)=[\mathrm{d}\Gamma(\partial'_l)p]\star q+p\star[\mathrm{d}\Gamma(\partial'_l)q]\tag{2}$$

for any $p,q\in\mathsf{P}(\mathcal{G}'_+)$ (similarly, for $\mathrm{d}\Gamma(\partial_l)$ on $\mathsf{P}'(\mathcal{G}'_+)$).

(iv) *The generators $\mathrm{d}\Gamma(\partial_l)$ and $\mathrm{d}\Gamma(\partial'_l)$ satisfy the relations*

$$\langle\mathrm{d}\Gamma(\partial_l)p\mid q\rangle=-\left\langle p\mid\mathrm{d}\Gamma(\partial'_l)q\right\rangle,\quad p\in\mathsf{P}'(\mathcal{G}'_+),\,q\in\mathsf{P}(\mathcal{G}'_+).$$

Proof. (i) From [12] it follows the topological isomorphism $\mathcal{G}_+\approx\otimes^d_{\mathrm{p}}\mathcal{G}(\mathbb{R}^1_+)$. Hence, $\odot^n_{\mathrm{p}}\mathcal{G}_+\approx\odot^n_{\mathrm{p}}[\otimes^d_{\mathrm{p}}\mathcal{G}(\mathbb{R}^1_+)]$.

The topology of $\mathcal{G}^{\mu_l}_{[0,b_l]}(\mathbb{R}^1_+) := \mathcal{G}^{\mu_l}_{[a_l,b_l]}(\mathbb{R}^1)/\{[\mathcal{G}'(\mathbb{R}^1_+)]^{\perp} \bigcap \mathcal{G}^{\mu_l}_{[a_l,b_l]}(\mathbb{R}^1)\}$ $(l = 1, \dots, d)$ is defined by the norms

$$\|\boldsymbol{\varphi}\|_{\mathcal{G}^{\mu_l}_{[0,b_l]}} = \sup_{\tau_l \in [a_l,b_l]} \sup_{k_l \in \mathbb{Z}_+} \frac{|\partial^{k_l} \boldsymbol{\varphi}(\tau_l)|}{\mu_l^{k_l} k_l^{\beta k_l}}.$$

The space $\mathcal{G}(\mathbb{R}^1_+) \approx \operatorname{ind\,lim}_{b_l,\mu_l \to \infty} \mathcal{G}^{\mu_l}_{[0,b_l]}(\mathbb{R}^1_+)$ is an inductive limit with the compact injections $\mathcal{G}^{\mu_l}_{[0,b_l]}(\mathbb{R}^1_+) \hookrightarrow \mathcal{G}^{\nu_l}_{[0,b'_l]}(\mathbb{R}^1_+)$, where $\mu_l < \nu_l$, $b_l < b'_l$ [11]. Using the known (see [9]) commutative property of inductive limits with projective tensor products as well as the continuity and openness of $\mathfrak{s}_n$, we obtain

$$\begin{aligned} \odot^n_{\mathfrak{p}} \mathcal{G}_+ &\approx \odot^n_{\mathfrak{p}} \left[\operatorname*{ind\,lim}_{b_1,\mu_1 \to \infty} \mathcal{G}^{\mu_1}_{[0,b_1]}(\mathbb{R}^1_+) \otimes_{\mathfrak{p}} \cdots \otimes_{\mathfrak{p}} \operatorname*{ind\,lim}_{b_d,\mu_d \to \infty} \mathcal{G}^{\mu_d}_{[0,b_d]}(\mathbb{R}^1_+) \right] \\ &\approx \operatorname*{ind\,lim}_{b,\mu \to \infty} \odot^n_{\mathfrak{p}} \left[\mathcal{G}^{\mu_1}_{[0,b_1]}(\mathbb{R}^1_+) \otimes_{\mathfrak{p}} \cdots \otimes_{\mathfrak{p}} \mathcal{G}^{\mu_d}_{[0,b_d]}(\mathbb{R}^1_+) \right]. \end{aligned} \tag{3}$$

Due to the polarization formula the set of functions

$$q_n \colon (\tau^1_1, \dots, \tau^1_d, \dots, \tau^n_1, \dots, \tau^n_d) \longmapsto \prod_{i=1}^n \left[\boldsymbol{\varphi}_1(\tau^i_1) \cdot \ldots \cdot \boldsymbol{\varphi}_d(\tau^i_d) \right], \tag{4}$$

with $\boldsymbol{\varphi}_l \in \mathcal{G}^{\mu_l}_{[0,b_l]}(\mathbb{R}^1_+)$ for any $l = 1, \dots, d$ and $b_l > 0$, is total in $\odot^n_{\mathfrak{p}} \mathcal{G}_+$.

Proposition 2.1 implies that we have

$$\left[\Upsilon_{\mathcal{G}_+} \Gamma(T'_{t_l}) \Upsilon^{-1}_{\mathcal{G}_+} \right] \mathrm{Q}(f) = \sum_{n \in \mathbb{Z}_+} \left\langle \otimes^n (T'_{t_l} f) \mid q_n \right\rangle = \sum_{n \in \mathbb{Z}_+} \left\langle \otimes^n f \mid (\otimes^n T_{t_l}) q_n \right\rangle$$

for any $q_n \in \odot^n_{\mathfrak{p}} \mathcal{G}_+$. Let us consider the semigroup $\otimes^n T_{t_l}$ on a total set (4). Since $\tau_l - t_l \in \operatorname{supp}(T_{t_l} \boldsymbol{\varphi}_l)$ iff $\tau_l \in \operatorname{supp} \boldsymbol{\varphi}_l$, thus

$$\operatorname{supp}(T_{t_l} \boldsymbol{\varphi}_l) = (\operatorname{supp} \boldsymbol{\varphi}_l - t_l) \bigcap [0, \infty) \quad \text{with } t_l \geq 0.$$

Hence, the following inequality

$$\|T_{t_l} \boldsymbol{\varphi}_l\|_{\mathcal{G}^{\mu_l}_{[0,b_l]}} \leq \|\boldsymbol{\varphi}_l\|_{\mathcal{G}^{\mu_l}_{[0,b_l]}} \quad \text{for any } \boldsymbol{\varphi}_l \in \mathcal{G}^{\mu_l}_{[0,b_l]}(\mathbb{R}^1_+),\ t_l \geq 0$$

is true.

Now, the regularity of inductive limit (3) implies that the semigroup $\otimes^n T_{t_l}$ is equibounded and, as a consequence, it is equicontinuous on $\odot^n_{\mathfrak{p}} \mathcal{G}_+$. The last conclusion uses barreledness of $\odot^n_{\mathfrak{p}} \mathcal{G}_+$ and the uniform boundedness Banach–Steinhaus principle. As a result, each semigroup $\otimes^n T_{t_l}$ is equicontinuous on $(\odot^n_{\mathfrak{p}} \mathcal{G}_+)$ for all n (see e.g. [15]).

The smoothness of $\partial^{k_l}_l T_{t_l} q_n$ with $k_l \in \mathbb{Z}_+$ by the variable $t_l \geq 0$ allows to apply the Lagrange theorem, that gives the C_0-property for the semigroup $\otimes^n T_{t_l}$ on $\odot^n_{\mathfrak{p}} \mathcal{G}_+$. Then the equicontinuity and C_0-property for the semigroup $\Gamma(T'_{t_l})$ on $\mathrm{P}(\mathcal{G}'_+) \approx \bigoplus_n (\odot^n_{\mathfrak{p}} \mathcal{G}_+)$ directly follows from properties of direct sum topology.

From the inequality $(k_l + 1)^{(k_l+1)\beta} \leq 2^{(k_l+1)\beta} k_l^{k_l \beta}$ it follows

$$\|\partial_l \boldsymbol{\varphi}_l\|_{\mathcal{G}^{\nu_l}_{[0,b_l]}} \leq \nu_l \sup_{k_l \in \mathbb{Z}_+} \sup_{\tau_l \in [0,b_l]} \frac{|\partial^{k_l+1}_l \boldsymbol{\varphi}_l(\tau_l)|}{(\nu_l 2^{-\beta})^{k_l+1} (k_l+1)^{(k_l+1)\beta}} \leq \nu_l \|\boldsymbol{\varphi}_l\|_{\mathcal{G}^{\mu_l}_{+}[0,b_l]}$$

with $\mu_l = \nu_l 2^{-\beta}$. So, ${}^n_j\partial_l \in \mathscr{L}(\odot^n_{\mathfrak{p}} \mathcal{G}_+)$ and

$$\partial_l \left(\otimes^n T_{t_l} \prod_{i=1}^n \left[\boldsymbol{\varphi}_1(\tau^i_1) \ldots \boldsymbol{\varphi}_d(\tau^i_d) \right] \right) = \sum_{j=1}^n \left(\otimes^n T_{t_l} \right) {}^n_j\partial_l \prod_{i=1}^n \left[\boldsymbol{\varphi}_1(\tau^i_1) \ldots \boldsymbol{\varphi}_d(\tau^i_d) \right].$$

It remains to apply Proposition 2.3(ii) about approximation of every element $q \in \bigoplus_n (\odot^n_{\mathfrak{p}} \mathcal{G}_+)$ by a linear span of elements (4).

The assertion (ii) follows from Proposition (i) by application of the duality $\left\langle \bigtimes_n (\odot^n_{\mathfrak{p}} \mathcal{G}'_+) \mid \bigoplus_n (\odot^n_{\mathfrak{p}} \mathcal{G}_+) \right\rangle$.

(iii) Let $d_f \mathrm{Q}(\partial'_l f)$ be the Fréchet derivative of the polynomial $\mathrm{Q} \in \mathscr{P}(\mathcal{G}'_+)$, which is calculated at the point $f \in \mathcal{G}'_+$ in the direction $\partial'_l f$.

The generator $\mathrm{d}\Gamma(\partial'_l)$ of $\Gamma(T'_l)$ satisfies the following equality $d_f \mathrm{Q}(\partial'_l f) = \left[\Upsilon_{\mathcal{G}_+} \mathrm{d}\Gamma(\partial'_l) \Upsilon^{-1}_{\mathcal{G}_+} \right] \mathrm{Q}(f)$, since

$$\begin{aligned} \left[\Upsilon_{\mathcal{G}_+} \mathrm{d}\Gamma(\partial'_l) \Upsilon^{-1}_{\mathcal{G}_+} \right] \mathrm{Q}(T'_t f) &= \frac{\mathrm{d}}{\mathrm{d}t_l} \mathrm{Q}(T'_{t_l} f) \\ &= d_{T'_{t_l} f} \mathrm{Q} \left(\frac{\mathrm{d}}{\mathrm{d}t_l} T'_{t_l} f \right) = d_{T'_{t_l} f} \mathrm{Q}(\partial'_l T_{t_l} f), \end{aligned}$$

$$\begin{aligned}\left[\Upsilon_{\mathcal{G}_+}\mathrm{d}\Gamma(\partial_l')\Upsilon_{\mathcal{G}_+}^{-1}\right]Q(f) &= \left[\Upsilon_{\mathcal{G}_+}\mathrm{d}\Gamma(\partial_l')\Upsilon_{\mathcal{G}_+}^{-1}\right]Q(T_{t_l}'f)|_{t_l=0}\\ &= d_{T_{t_l}'f}Q(\partial_l T_{t_l}'f)|_{t_l=0} = d_f Q(\partial_l' f).\end{aligned}$$

The Leibniz property

$$\left[\Upsilon_{\mathcal{G}_+}\mathrm{d}\Gamma(\partial_l')\Upsilon_{\mathcal{G}_+}^{-1}\right](P\cdot Q)(f) = \left(\left[\Upsilon_{\mathcal{G}_+}\mathrm{d}\Gamma(\partial_l')\Upsilon_{\mathcal{G}_+}^{-1}\right]P\cdot Q\right)(f) + \left(P\cdot\left[\Upsilon_{\mathcal{G}_+}\mathrm{d}\Gamma(\partial_l')\Upsilon_{\mathcal{G}_+}^{-1}\right]Q\right)(f)$$

for all $P, Q \in \mathcal{P}(\mathcal{G}_+')$ arises from the formulae above. Now (2) follows from Proposition 2.4(i). For $\mathrm{d}\Gamma(\partial_l)$ similarly.

The assertion (iv) immediately follows from the dual relations

$$\partial_l' = -\partial_l, \qquad \left\langle p_n \,\middle|\, \sum_{j=1}^{n}{}_j^n\partial_l q_n\right\rangle = -\left\langle \sum_{j=1}^{n}{}_j^n\partial_l' p_n \,\middle|\, q_n\right\rangle$$

with $q_n \in \odot_{\mathfrak{p}}^n \mathcal{G}_+$ and $p_n \in \odot_{\mathfrak{p}}^n \mathcal{G}_+'$. □

Corollary 3.2. (i) *The* 1*-parameter families* $\mathcal{T}_l'\colon 0 \le t_l \longmapsto \mathcal{T}_{t_l}'$ *of algebraic automorphisms on multiplication algebra* $\mathcal{P}(\mathcal{G}_+')$*, which are defined as*

$$\mathcal{T}_{t_l}'Q(f) = Q(T_{t_l}'f), \quad Q \in \mathcal{P}(\mathcal{G}_+'), f \in \mathcal{G}_+',$$

are equicontinuous C_0*-semigroups with the generators*

$$d_f Q(\partial_l' f) = \left[\Upsilon_{\mathcal{G}_+}\mathrm{d}\Gamma(\partial_l')\Upsilon_{\mathcal{G}_+}^{-1}\right]Q(f).$$

(ii) *The* 1*-parameter families* $\mathcal{T}_l\colon 0 \le t_l \longmapsto \mathcal{T}_{t_l}$ *of algebraic automorphisms on the multiplicative algebra* $\mathcal{P}'(\mathcal{G}_+')$*, which are defined by the formula*

$$\mathcal{T}_{t_l}P(\boldsymbol{\varphi}) = P(T_{t_l}\boldsymbol{\varphi}), \quad P \in \mathcal{P}'(\mathcal{G}_+'),\ \boldsymbol{\varphi} \in \mathcal{G}_+$$

are equicontinuous C_0*-semigroups with the generators*

$$d_f P(\partial_l\boldsymbol{\varphi}) = \left[\Upsilon_{\mathcal{G}_+'}\mathrm{d}\Gamma(\partial_l)\Upsilon_{\mathcal{G}_+'}^{-1}\right]P(\boldsymbol{\varphi}).$$

Remark 3.3. We can call elements of $\mathcal{P}'(\mathcal{G}_+')$ *polynomial ultradistributions* on $\mathbb{R}_+^d$. Since $\mathcal{G}_+' \subset \mathcal{P}'(\mathcal{G}_+')$, elements of $\mathcal{G}_+'$ can be understood as linear ultradistributions.

4. A cross-correlation

We denote the tensor product of semigroups T_{t_l} by

$$T_t = T_{t_1} \otimes \cdots \otimes T_{t_d}, \quad t = (t_1, \ldots, t_d) \in \mathbb{R}_+^d.$$

Clearly, $T_+\colon \mathbb{R}_+^d \ni t \longmapsto T_t \in \mathcal{L}(\mathcal{G}_+)$ is an equicontinuous d-parameter C_0-semigroup. Let $T_+'\colon \mathbb{R}_+^d \ni t \longmapsto T_t' \in \mathcal{L}(\mathcal{G}_+')$ be the adjoint d-parameter semigroup with respect to the duality $\langle \mathcal{G}_+' \mid \mathcal{G}_+\rangle$.

Via Theorem 3.1 we can define the map $\Gamma(T_+')q\colon \mathbb{R}_+^d \ni t \longmapsto \Gamma(T_t')q \in \mathsf{P}(\mathcal{G}_+')$ such that for any $q = \sum_{n\in\mathbb{Z}_+} q_n \in \mathsf{P}(\mathcal{G}_+')$ with $q_n \in \mathsf{P}_n(\mathcal{G}_+')$ and $n \in \mathbb{Z}_+$

$$\Gamma(T_+')q_n\colon \mathbb{R}_+^d \ni t \longmapsto (\otimes^n T_t)q_n.$$

Let us approach q_n by linear combinations of elements (4). Then apply the known fact (see [12]) that any Gevrey smooth function with values in a nuclear space belongs to the complete injective tensor product. Finally we will obtain that the $\mathsf{P}_n(\mathcal{G}_+')$-value function $\partial^k T_t q_n$, $(k \in \mathbb{Z}_+^n)$ of the variable $t \in \mathbb{R}_+^d$ belongs to $\mathsf{P}_n(\mathcal{G}_+') \otimes_{\mathfrak{e}} \mathcal{G}_+$. Via the nuclear property, we have $\mathsf{P}_n(\mathcal{G}_+') \otimes_{\mathfrak{e}} \mathcal{G}_+ \approx \mathsf{P}_n(\mathcal{G}_+') \otimes_{\mathfrak{p}} \mathcal{G}_+$. Thus,

$$\Gamma(T_+')q \in \bigoplus_{n\in\mathbb{Z}_+}\left[\mathsf{P}_n(\mathcal{G}_+') \otimes_{\mathfrak{p}} \mathcal{G}_+\right].$$

Theorem 4.1. *The mapping, called the cross-correlation,*

$$\mathsf{K}\colon \mathcal{G}_+' \ni f \longmapsto f\circledast \in \mathcal{L}_\Gamma\left[\mathsf{P}(\mathcal{G}_+')\right], \qquad f \circledast q := \langle f \mid \Gamma(T_+')q\rangle \in \mathsf{P}(\mathcal{G}_+')$$

is an algebraic topological isomorphism from the convolution algebra $\mathcal{G}_+'$ *onto the commutant* $[[\Gamma(T_+')]]_\Gamma$ *of the group* $\Gamma(T_+')$ *in* $\mathcal{L}_\Gamma[\mathsf{P}(\mathcal{G}_+')]$ *and has the properties*

$$(f*g)\circledast q=f\circledast(g\circledast q),$$
$$\mathrm{d}\Gamma(\partial_l')(f\circledast q)=f\circledast\mathrm{d}\Gamma(\partial_l')q=\mathrm{d}\Gamma(\partial_l)f\circledast q,$$
$$\mathrm{d}\Gamma(\partial_l')=\partial_l\delta\circledast$$

with $f,g\in\mathcal{G}_+'$ *and* $l=1,\dots,d$. *Moreover, the identity operator in the space* $\mathscr{L}_\Gamma[\mathsf{P}(\mathcal{G}_+')]$ *has the form* $(\otimes^d\delta)\circledast$.

Proof. For any $q_n\in\odot_{\mathfrak{p}}^n\mathcal{G}_+$ and $f\in\mathcal{G}_+'$ we have $\langle f\mid(\otimes^nT_+)q_n\rangle\in\odot_{\mathfrak{p}}^n\mathcal{G}_+$, since $(\otimes^nT_+)q_n\in\odot_{\mathfrak{p}}^n\mathcal{G}_+\otimes_{\mathfrak{p}}\mathcal{G}_+$. Each narrowed mapping

$$\mathsf{K}_n\colon\mathcal{G}_+'\ni f\longrightarrow f\circledast|_{\mathsf{P}_n(\mathcal{G}_+')}\in\mathscr{L}\left[\mathsf{P}_n(\mathcal{G}_+')\right]$$

with $f\circledast q_n=\langle f\mid(\otimes^nT_+)q_n\rangle$ is injective (in view of injectivity of T_+) and it acts as an algebraic isomorphism. In fact, the convolution in $\mathcal{G}_+'$ can be defined by the duality $\langle\mathcal{G}'(\mathbb{R}^d)\mid\mathcal{G}(\mathbb{R}^d)\rangle$ as follows

$$\langle f*g\mid\varphi\rangle=\langle f(t)\mid\xi(t)\,\langle g(s)\mid\eta(s)\varphi(t+s)\rangle\rangle$$

for any $\varphi\in\mathcal{G}(\mathbb{R}^d)$, where the functions $\xi,\eta\in\mathcal{G}(\mathbb{R}^d)$ are equal to 1 on supp φ and to 0 outside of a neighborhood of supp φ (see e.g. [14,16]). Then we obtain

$$\begin{aligned}(f*g)\circledast q_n&=\left\langle f(t)\mid\xi(t)\left\langle g(s)\mid\eta(s)(\otimes^nT_{t+s})q_n\right\rangle\right\rangle\\&=\left\langle f(t)\mid\xi(t)g\circledast\left[\eta(s)(\otimes^nT_{t+s})q_n\right]\right\rangle=f\circledast(g\circledast q_n).\end{aligned}$$

As a consequence, $(\otimes^d\delta)\circledast$ is the unit in $\mathscr{L}[\mathsf{P}_n(\mathcal{G}_+')]$. It follows that

$$\begin{aligned}\partial_l\delta\circledast q_n&=\langle\partial_l\delta\mid(\otimes^nT_+)q_n\rangle\\&=-\mathrm{i}\frac{\partial}{\partial t_l}\left(\otimes^nT_{t_1,\dots,t_d}\right)q_n|_{t=0}=-\sum_{j=1}^n{}_j^n\partial_lq_n.\end{aligned}$$

Via Theorem 3.1 we obtain $\partial_l\delta\circledast q=\mathrm{d}\Gamma(\partial_l')q$. Replacing q_n by $\sum_j^n\partial_lq_n$ in the expression $f\circledast q_n=\langle f\mid(\otimes^nT_+)q_n\rangle$ we obtain

$$\begin{aligned}f\circledast\left(\sum\nolimits_j^n\partial_l\right)q_n&=\left\langle f\left|\sum\nolimits_j^n\partial_l\ (\otimes^nT_+)q_n\right.\right\rangle\\&=\sum\nolimits_j^n\partial_l\left\langle f\mid(\otimes^nT_+)q_n\right\rangle=\sum\nolimits_j^n\partial_l(f\circledast q_n).\end{aligned}$$

Hence, $f\circledast|_{\mathsf{P}_n(\mathcal{G}_+')}\in[[\sum_j^n\partial_l]]$, $\forall l=1,\dots,d$, $\forall n$, i.e., $f\circledast\in[[\Gamma(T_+')]]_\Gamma$.

Now we will prove that the codomain of K_n equals the commutant $[[\otimes^nT_t]]$ in $\mathscr{L}[\mathsf{P}_n(\mathcal{G}_+')]$. Let $K\in\mathscr{L}[\mathsf{P}_n(\mathcal{G}_+')]$ be an operator for which

$$\left[K(\otimes^nT_t)\right]q_n=\left[(\otimes^nT_t)K\right]q_n.$$

Let us show, that there exists a functional $f\in\mathcal{G}_+'$ such that $K=f\circledast|_{\mathsf{P}_n(\mathcal{G}_+')}$. The functional

$$\langle f\mid q_n\rangle:=(Kq_n)(0),\quad q_n\in\odot_{\mathfrak{p}}^n\mathcal{G}_+$$

is required. In fact, putting $\otimes^nT_tq_n$ instead of q_n, we obtain

$$\begin{aligned}(f\circledast q_n)(s)&=\langle f(t)\mid\otimes^nT_tq_n(s)\rangle=\langle f(t)\mid\otimes^nT_sq_n(t)\rangle\\&=\left[K(\otimes^nT_sq_n)\right](0)=Kq_n(s)\quad\text{with } s\in\mathbb{R}_+^d.\end{aligned}$$

Using an arbitrariness of n, we conclude that a codomain of the matrix diagonal algebraic homomorphism $\mathsf{K}=\left[\begin{Bmatrix}\mathsf{K}_n&:&n=k\\0&:&n\neq k\end{Bmatrix}\right]_{n,k\in\mathbb{Z}_+}$ coincides with the commutant $[[\Gamma(T_+')]]_\Gamma$ of the group $\Gamma(T_+')=\left[\begin{Bmatrix}\otimes^nT_t&:&n=k\\0&:&n\neq k\end{Bmatrix}\right]_{n,k\in\mathbb{Z}_+}$. From construction of each mapping K_n it follows that they are nuclear [7]. Hence, K is continuous and has a closed codomain, since it coincides with the commutant. Therefore, the open mapping Banach theorem implies that K is a topological algebraic isomorphism.

Finally, since

$$(\partial_l\delta*f)*q=\partial_lf*q=f*\partial_lq=f*(\partial_l\delta*q),$$

for any polynomial $q\in\mathsf{P}(\mathcal{G}_+')$ we have

$$\begin{aligned}\mathrm{d}\Gamma(\partial_l')(f\circledast q)&=\partial_l\delta\circledast(f\circledast q)=(\partial_l\delta*f)\circledast q\\&=[\mathrm{d}\Gamma(\partial_l)f]\circledast q=f\circledast(\partial_l\delta*q)=f\circledast\mathrm{d}\Gamma(\partial_l')q\end{aligned}$$

and the theorem is proved completely. □

5. A polynomially extended Laplace transformation

According to the Paley–Wiener theorem the Fourier transformation

$$\widehat{\varphi}(\zeta) := (F\varphi)(\zeta) = \int e^{-i(t,\zeta)}\varphi(t)dt \quad \text{with } \varphi \in \mathcal{G}(\mathbb{R}^d),\ \zeta \in \mathbb{C}^d,\ t \in \mathbb{R}^d,$$

is a topological isomorphism $F: \mathcal{G}(\mathbb{R}^d) \longmapsto \widehat{\mathcal{G}}(\mathbb{C}^d)$ onto a space $\widehat{\mathcal{G}}(\mathbb{C}^d)$ of entire analytic functions endowed for simplicity with the inductive LC topology, generated by F. Here $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{C}^d$. Let

$$\widehat{\mathcal{G}}_+ := \widehat{\mathcal{G}}(\mathbb{C}^d)/F\left[[\mathcal{G}'(\mathbb{R}^d_+)]^\perp\right]$$

stand for the corresponding LC factor space. For the strong duals the appropriate adjoint transformation $F': \widehat{\mathcal{G}}'(\mathbb{C}^d) \longmapsto \mathcal{G}'(\mathbb{R}^d)$ is defined. The codomain

$$\widehat{\mathcal{G}}'_+ := F'^{-1}(\mathcal{G}'_+) \quad \text{of the subspace } \mathcal{G}'_+ \subset \mathcal{G}'(\mathbb{R}^d)$$

with respect to the inverse mapping $F'^{-1}: \mathcal{G}'(\mathbb{R}^d) \ni f \longmapsto \widehat{f} \in \widehat{\mathcal{G}}'(\mathbb{C}^d)$ is closed in the dual $\widehat{\mathcal{G}}'(\mathbb{C}^d)$. The mappings F' and F'^{-1} are continuous with respect to the appropriate strong topologies. It follows that $\widehat{\mathcal{G}}'_+$ is a nuclear (F) space. The space $\widehat{\mathcal{G}}'_+$ is a multiplicative topological algebra with the unit $\otimes^d\widehat{\delta}$, since

$$\widehat{(f * g)} = \widehat{f}\cdot\widehat{g}, \quad f, g \in \mathcal{G}'_+.$$

A generalized Laplace transformation can be defined as

$$F'_+: \mathcal{G}'_+ \ni f \longmapsto \widehat{f} \in \widehat{\mathcal{G}}'_+, \qquad F'_+ := F'^{-1}|_{\mathcal{G}'_+}. \tag{5}$$

Any element of $\widehat{\mathcal{G}}_+$ can be interpreted, as the Laplace transform $\widehat{\boldsymbol{\varphi}} = F'_+(\boldsymbol{\varphi})$ of regular ultradistribution

$$\boldsymbol{\varphi} := \Theta(\varphi) \in \mathcal{G}'_+ \quad \text{with } \varphi \in \mathcal{G}(\mathbb{R}^d).$$

Therefore, the Laplace transformation of $\mathcal{G}_+ = \mathcal{G}(\mathbb{R}^d)/[\mathcal{G}'(\mathbb{R}^d_+)]^\perp$ is a restriction of F'_+. From duality arguments it follows that the topological surjective isomorphism

$$F_+: \mathcal{G}_+ \ni \boldsymbol{\varphi} \longmapsto \widehat{\boldsymbol{\varphi}} \in \widehat{\mathcal{G}}_+, \qquad F_+ := F'_+|_{\mathcal{G}_+}$$

is true and has the form (below $\zeta = (\zeta_1, \dots, \zeta_d)$)

$$\widehat{\boldsymbol{\varphi}}(\zeta) = \int_{\mathbb{R}^d_+} e^{-i(t,\zeta)}\boldsymbol{\varphi}(t)dt \quad \text{with } \boldsymbol{\varphi} \in \mathcal{G}_+,\ \operatorname{Im}\zeta_1 \le 0, \dots, \operatorname{Im}\zeta_d \le 0.$$

Function $\widehat{\boldsymbol{\varphi}}(\zeta)$ is analytic in a tube complex domain [14]. It is not difficult to calculate that

$$\widehat{\partial_l\boldsymbol{\varphi}}(\zeta) = \zeta_l\widehat{\boldsymbol{\varphi}}(\zeta) - \boldsymbol{\varphi}(0), \quad \boldsymbol{\varphi} \in \mathcal{G}_+.$$

It also follows that $\widehat{\mathcal{G}}_+$ is a nuclear (DF) space.

Applying Proposition 2.1 we can extend the generalized Laplace transformation (5) onto the algebra $\mathcal{P}'(\mathcal{G}'_+)$ as follows.

Proposition 5.1. *The commutative diagrams*

$$\begin{array}{ccc} \mathcal{P}_n(\mathcal{G}_+) & \xrightarrow{\mathcal{F}'_n} & \mathcal{P}_n(\widehat{\mathcal{G}}_+) \\ \Upsilon_n^{\mathcal{G}'_+}\Big\| & & \Upsilon_n^{\widehat{\mathcal{G}}'_+}\Big\| \\ \odot^n_{\mathrm{p}}\mathcal{G}'_+ & \xrightarrow{\otimes^n F'_+} & \odot^n_{\mathrm{p}}\widehat{\mathcal{G}}'_+ \end{array}, \qquad \begin{array}{ccc} \mathcal{P}'(\mathcal{G}'_+) & \xrightarrow{\mathcal{F}'_+} & \mathcal{P}'(\widehat{\mathcal{G}}'_+) \\ \Upsilon_{\mathcal{G}'_+}\Big\| & & \Upsilon_{\widehat{\mathcal{G}}'_+}\Big\| \\ \underset{n\in\mathbb{Z}_+}{\times}(\odot^n_{\mathrm{p}}\mathcal{G}'_+) & \xrightarrow{\Gamma(F'_+)} & \underset{n\in\mathbb{Z}_+}{\times}(\odot^n_{\mathrm{p}}\widehat{\mathcal{G}}'_+), \end{array}$$

in homogeneous and general cases respectively, uniquely define the polynomial extension

$$\mathcal{F}'_+: \mathcal{P}'(\mathcal{G}'_+) \ni P = \times_{n\in\mathbb{Z}_+}P_n \longmapsto \widehat{P} := \times_{n\in\mathbb{Z}_+}\mathcal{F}'_n(P_n) \in \mathcal{P}'(\widehat{\mathcal{G}}'_+), \quad P_n \in \mathcal{P}_n(\mathcal{G}_+)$$

of the generalized Laplace transformation F'_+. Let us note that $\mathcal{F}'_+$ has the matrix diagonal form

$$\begin{aligned}\mathcal{F}'_+ &= \left[\begin{cases}\mathcal{F}'_n & : n = k\\ 0 & : n \ne k\end{cases}\right]_{n,k\in\mathbb{Z}_+} \in \mathcal{L}_\Gamma\left[\mathcal{P}'(\mathcal{G}'_+), \mathcal{P}'(\widehat{\mathcal{G}}'_+)\right]\\ &:= \mathcal{L}\left[\mathcal{P}'(\mathcal{G}'_+), \mathcal{P}'(\widehat{\mathcal{G}}'_+)\right]\bigcap\left[\begin{cases}\mathcal{L}\left[\mathcal{P}_n(\mathcal{G}_+), \mathcal{P}_n(\widehat{\mathcal{G}}_+)\right] & : n = k\\ 0 & : n \ne k\end{cases}\right]_{n,k\in\mathbb{Z}_+}\end{aligned}$$

with $\mathcal{F}'_n \in \mathcal{L}[\mathcal{P}_n(\mathcal{G}_+), \mathcal{P}_n(\widehat{\mathcal{G}}_+)]$. Moreover, $\mathcal{F}'_+$ is invariant with respect to the polynomial multiplication and acts as an algebraic surjective topological isomorphism from $\mathcal{P}'(\mathcal{G}'_+)$ onto $\mathcal{P}'(\widehat{\mathcal{G}}'_+)$.

Note that the assertion about algebraic isomorphism is a direct consequence of the previous diagrams and Proposition 2.3.

Propositions 2.1 and 2.4 imply that the restrictions

$$\mathcal{F}_+ := \mathcal{F}'_+|_{\mathcal{P}(\mathcal{G}'_+)}, \qquad \Gamma(F_+) := \Gamma(F'_+)|_{\mathsf{P}(\mathcal{G}'_+)}$$

to the dense subalgebras $\mathcal{P}(\mathcal{G}'_+) \subset \mathcal{P}'(\mathcal{G}'_+)$ and $\mathsf{P}(\mathcal{G}'_+) \subset \mathsf{P}'(\mathcal{G}'_+)$, respectively, act as the algebraic isomorphisms

$$\mathcal{F}_+ \colon \mathcal{P}(\mathcal{G}'_+) \ni Q = \sum_{n\in\mathbb{Z}_+} Q_n \longrightarrow \widehat{Q} := \sum_{n\in\mathbb{Z}_+} \mathcal{F}_n(Q_n) \in \mathcal{P}(\widehat{\mathcal{G}}'_+), \quad Q_n \in \mathcal{P}_n(\mathcal{G}'_+)$$

$$\Gamma(F_+) \colon \mathsf{P}(\mathcal{G}'_+) \ni q = \sum_{n\in\mathbb{Z}_+} q_n \longrightarrow \widehat{q} := \sum_{n\in\mathbb{Z}_+} \widehat{q}_n \in \mathsf{P}(\widehat{\mathcal{G}}'_+), \quad q_n \in \mathsf{P}_n(\mathcal{G}'_+)$$

with $\mathcal{F}_n := \mathcal{F}'_n|_{\mathcal{P}_n(\mathcal{G}'_+)}, \widehat{q}_n := \otimes^n F_n(q_n)$ and the following dualities

$$\left\langle \widehat{P} \mid \widehat{Q} \right\rangle = \langle P \mid Q \rangle, \qquad \langle \widehat{p} \mid \widehat{q} \rangle = \langle p \mid q \rangle$$

are true.

The next corollary follows from Proposition 5.1 and Theorem 3.1.

Corollary 5.2. *The family* $\Gamma(\widehat{T'_+}) \colon \mathbb{R}^d_+ \ni t \longmapsto \Gamma(\widehat{T'_t}) \in \mathscr{L}_\Gamma[\mathsf{P}(\widehat{\mathcal{G}'_+})]$ *acting as*

$$\Gamma(\widehat{T'_t})\widehat{q}_n := \otimes^n F_+ \left[(\otimes^n T_t)q_n\right], \qquad \Gamma(\widehat{T'_t})\widehat{q} = \sum_{n\in\mathbb{Z}_+} \Gamma(\widehat{T'_t})\widehat{q}_n$$

for any $q = \sum_{n\in\mathbb{Z}_+} q_n \in \mathsf{P}(\mathcal{G}'_+)$ *with* $q_n \in \odot^n_{\mathfrak{p}} \mathcal{G}_+$ *is an equicontinuous* C_0*-semigroup of automorphisms on* $\mathsf{P}(\widehat{\mathcal{G}}'_+)$ *with the generators*

$$\mathsf{d}\Gamma(\widehat{\partial}'_l) = \left[\left[\begin{cases} \sum_{j=1}^{n} {}^n_j\widehat{\partial}_l & : n = k \\ 0 & : n \neq k \end{cases}\right]_{n,k\in\mathbb{Z}_+}\right] \in \mathscr{L}_\Gamma\left[\mathsf{P}(\widehat{\mathcal{G}}'_+)\right],$$

where ${}^n_j\widehat{\partial}_l := \otimes^{j-1}\widehat{1}_+ \otimes \widehat{\partial}_l \otimes^{n-j} \widehat{1}_+$, $l = 1, \ldots, d$ *and* $\widehat{1}_+$ *denotes the identity operator in* $\mathscr{L}(\widehat{\mathcal{G}}_+)$.

Now for any $f \in \mathcal{G}'_+$ we can uniquely define a linear operator

$$(\widehat{\mathsf{K}f})\widehat{q} := \widehat{f \circledast q} \in \mathsf{P}(\widehat{\mathcal{G}}'_+), \qquad f \circledast q = \left\langle f \mid \Gamma(T'_+)q \right\rangle \in \mathsf{P}(\mathcal{G}'_+), \quad q \in \mathsf{P}(\mathcal{G}'_+),$$

generated by the polynomially extended Laplace transformation and the cross-correlation.

Proposition 5.3. *The operator* $\widehat{\mathsf{K}}$ *is an algebraic topological isomorphism from the multiplicative algebra* $\widehat{\mathcal{G}}'_+$ *onto the commutant* $[[\Gamma(\widehat{T'_+})]]_\Gamma$ *of the group* $\Gamma(\widehat{T'_+})$ *on* $\mathsf{P}(\widehat{\mathcal{G}}'_+)$ *and possesses the following properties:*

$$\widehat{\mathsf{K}\widehat{\delta}} \text{ is the unit in } \mathscr{L}_\Gamma\left[\mathsf{P}(\widehat{\mathcal{G}}'_+)\right] \quad \text{and} \quad \widehat{\partial_l\delta} \longmapsto \mathsf{d}\Gamma(\widehat{\partial}'_l), \quad l = 1, \ldots, d,$$

$$\widehat{f} \cdot \widehat{g} \longmapsto \left(\widehat{\mathsf{K}f}\right) \circ \left(\widehat{\mathsf{K}g}\right) \quad \text{for all } \widehat{f}, \widehat{g} \in \widehat{\mathcal{G}}'_+,$$

where $\circ$ *denotes the composition in* $\mathscr{L}_\Gamma[\mathsf{P}(\widehat{\mathcal{G}}'_+)]$.

Proof. The statement is a corollary of Theorem 4.1 and Corollary 5.2. □

Remark 5.4. Note that, the following commutative diagram

$$\begin{array}{ccc} \mathcal{G}'_+ & \xrightarrow{\ \mathsf{K}\ } & [[\Gamma(T'_+)]]_\Gamma \\ {\scriptstyle F'_+}\big\downarrow & & \big\downarrow \\ \widehat{\mathcal{G}}'_+ & \xrightarrow{\ \widehat{\mathsf{K}}\ } & \left[\left[\Gamma(\widehat{T'_+})\right]\right]_\Gamma \end{array}$$

defines a topological algebraic isomorphism of the corresponding commutants.

References

[1] H. Borchers, Algebras of unbounded operators in quantum fields theory, Physica 124A (1988) 1–127.
[2] T. Hida, H.H. Kuo, J. Potthoff, L. Streit, White Noise: An Infinite Dimensional Calculus, Kluwer Academic Publishers, 1993.
[3] Y.G. Kondratiev, L. Streit, W. Westerkamp, J. Yan, Generalized functions in infinite dimensional analysis, Hiroshima Math. J. 28 (1998) 213–260.
[4] N. Obata, White Noise Calculus and Fock Space, Springer, 1995.

[5] P.J. Boland, S. Dineen, Holomorphy on spaces of distribution, Pacific J. Math. 92 (1) (1981) 27–34.
[6] S. Dineen, Complex Analysis on Infinite Dimensional Spaces, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1999.
[7] A. Pietsch, Nukleare Localkonvexe Räume, Springer, 1965.
[8] H. Schaefer, Topological Vector Spaces, Macmillan, New York, 1971.
[9] A. Grothendieck, Produits tensoriels topologues et espases nucléaire, Mem. Amer. Math. Soc. 16 (11) (1955) 1–140.
[10] H. Komatsu, An Introduction to the Theory of Generalized Functions, Tokyo University Publ., 2000.
[11] J.L. Lions, E. Magenes, Problemes aux Limites Non Homogenes et Applications, vol. 3, Dunod, Paris, 1970.
[12] B.S. Mityagin, Nucl̀érité et autres propriétés des espaces de type *s*, Trudy Moscow Math. Sci. 9 (1960) 317–328.
[13] C. Roumieu, Sur quelques extentions de la notion de distribution, Ann. Sci. Éc. Norm. Supér. Paris. 77 (1960) 47–121.
[14] R.D. Carmichael, A. Kamiński, S. Pilipović, Boundary Values and Convolution in Ultradistribution Spaces, World Scientific, Singapore, 2007.
[15] K. Yosida, Functional Analysis, Springer-Verlag, New York, Berlin, Heidelberg, 1965.
[16] V.S. Vladimirov, Generalized Functions in Mathematical Physics, Nauka, Moscow, 1979.
[17] H. Jarchow, Locally Convex Spaces, Teubner, Stuttgart, 1981.